OPTIMAL CONTROL OF LINEAR ECONOMETRIC SYSTEMS WITH FINITE TIME HORIZON*

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1. THE PROBLEM

Given a linear econometric model in its reduced form

\[ y_t = Ay_{t-1} + Cx_t + b + u_t, \]

where \( y_t \) is a vector of \( p \) dependent variables, \( x_t \) is a vector of \( q \) variables subject to control, and where \( Eu_t = 0 \), \( Eu_t'u_t' = V \) and \( Eu_t'u_s' = 0 \) for \( t \neq s \), and given a quadratic welfare function for \( T \) periods

\[ W = E \sum_{t=1}^{T} (-y_t'K_t y_t + 2k_t'y_t), \]

\( K_t \) being symmetric and positive semi-definite, an important, and familiar, problem is to find \( x_1, \ldots, x_T \) to maximize expected welfare \( W \), the expectation being conditional on the given initial condition \( y_0 \). However, this problem can be formulated in different ways, two of which will now be stated. The solutions obtained by these formulations will be different and will serve different purposes.

The formulation familiar to economists is the one due to Simon [5], and adopted to linear econometric systems by Theil [6]. It requires expressing the econometric model in final form, i.e.,

\[ y_1 = Cx_1 + (Ay_0 + b + u_1) = R_{11}x_1 + s_1 \]
\[ y_2 = ACx_1 + Cx_2 + (As_1 + b + u_2) = R_{21}x_1 + R_{22}x_2 + s_2 \]
\[ y_3 = A^2Cx_1 + ACx_2 + Cx_3 + (As_2 + b + u_3) \]
\[ = R_{31}x_1 + R_{32}x_2 + R_{33}x_3 + s_3, \]
where the notations \( R_{ij} \) and \( s_j \) are Theil's [6]. It treats each \( x_t \) 
\((t=1,\ldots,T)\) as a function of past \( s_1,\ldots,s_{t-1} \) and the parameters in 
the conditional distribution of the future \( s_t,\ldots,s_T \). Its main result 
is concerned with the optimal \( x_1 \) for the first period. This optimal 
\( x_1 \) is expressed as a linear function of the expectations \( E s_1,\ldots,E s_T \) 
of all future random vectors \( s_t \) in (1.3). If the economic model 
were deterministic, with \( u_t \) in (1.1) being zero, i.e., if \( s_t \) in 
(1.3), which are functions of \( u_1,\ldots,u_t \), were non-stochastic, the 
optimal \( x_1 \) for the maximization of (1.2) would be the same linear 
function of \( s_1,\ldots,s_T \) as the above linear function of \( E s_1,\ldots,E s_T \). 
This result is the well known first-period certainty equivalence.

A second formulation of the maximization problem, the one chosen 
in this paper, is to retain the reduced form (1.1) and to express each 
optimal \( x_t \) as a function of \( Y_{t-1} \). It seems natural to ask how the 
control variables \( x_t \), such as government expenditures and money 
supply, should respond to recent observations \( Y_{t-1} \) on economic vari-
ables such as GNP, employment, and the price level. It seems less 
natural to express the optimal \( x_t \) as a function of the (not directly 
observed) variables \( s_1,\ldots,s_{t-1} \) and \( E s_1,\ldots,E s_T \).

In the solution, we obtain simultaneously the optimal feedback 
relations between \( x_t \) and \( Y_{t-1} \) for all periods from 1 to T, and 
not just the optimal \( x_1 \) for the first period. True, in the beginning 
of period 1, the decision maker does not have to act on \( x_2 \) (if his 
decision can be implemented without delay), and he can do better by 
waiting till the beginning of period 2 when more information shall
have become available. Yet, if the econometric model (1.1) remains valid, all useful additional information for period 2 will be contained in $y_1$ (since $y_2$ depends only on $y_1$, $x_2$, and the random vector $u_t$ with a known distribution); our solution tells how he should respond optimally to additional information in his future decisions. Thus the problem is solved for all periods.

If an economic decision maker were willing to take an econometric model (and an accompanying welfare function) so seriously as to accept the maximizing $x_t$ for each period without questions, then the difference between the above two approaches would reduce simply to a difference in methods of computation. In the first approach, a T-period problem would be solved only to yield the solution for the first period. In the second approach, the T-period problem would be solved to yield answers for future periods - until, of course, the econometric model or the welfare function is revised. Insofar as the decision maker has to be convinced of the validity of the optimal solution, the second form is easier to communicate to him since he is likely to think in terms of reacting to recent observations contained in $Y_{t-1}$. Furthermore, even if existing econometric models are not accurate enough for fine tuning, they may be used, in conjunction with the second approach, to indicate possibly desirable patterns of policy responses to recent economic events, and thus also to evaluate the observed patterns of responses by actual decision makers. It is for these applications that our formulation is intended.
Our formulation of the problem is certainly not new - feedback control based on observations on the state variables is a familiar concept in the literature of control theory, e.g., [7], [8], and [9]. Neither do we claim that the solution to optimal feedback control with a linear stochastic model and a quadratic welfare function is new - it is in fact well-known in control theory - although, perhaps, we have stated the problem and expressed the solution in a simpler and more convenient form than are often found in the literature. However, we do claim, first, that the solution to this problem by the elementary technique of Lagrange multipliers given in Sections 2 and 3 is new. Second, although dynamic programming is a technique commonly used for control problems, the exposition in Section 4 is simpler and more transparent than what the author can find in the literature. Third, we have also applied the method of Lagrange multipliers to solve the optimal control problem when there are delays in obtaining information or in carrying out decisions, thus providing a method for evaluating the cost of delays in terms of the change in expected welfare.

Partly to achieve simplicity, we have considered only linear feedback control equations, i.e.,

\[ x_t = G_t y_{t-1} + g_t \]  

and posed the problem of maximizing \( W \) with respect to the matrices \( G_t \) and the vectors \( g_t \) (\( t=1, \ldots, T \)). We believe that the solution to linear feedback control is itself important, for the purposes of communicating the result to decision makers and of evaluating the
observed patterns of policy responses to recent economic data. In
other words, even if the optimal feedback equation is non-linear, the
optimal linear feedback equation is of interest. It so happens that
the optimal feedback equation is linear for a linear model and a
quadratic welfare function, as is well-known in the control literature,
e.g. [9], but we do not feel apologetic for not dealing with this
mathematical point in the present paper.

In a previous paper [3], the steady-state solution to the
problem was obtained by the elementary method of Lagrange multipliers.
When a steady state is reached, we assume expected welfare to be

\[(1.5) \quad W = E(-y_t^x K y_t + 2k'y_t)\]

and the feedback control equation to be

\[(1.6) \quad x_t = G y_{t-1} + g\]

for all \( t \). A steady-state solution could be obtained only by
assuming, first, that the parameters \( K \) and \( k \) in the welfare
function (1.2) are unchanging through time and, second, that the sys-
tem with suitable control, i.e., the system (1.1) with (1.6) substituted,
is covariance-stationary. By treating a finite time horizon, this
paper relaxes both of these assumptions, and will serve as a continua-
tion of the previous paper [3].

We will find it convenient to decompose \( y_t \) into its mean
\( E y_t \) and deviation from mean \( y_t^x \),

\[(1.7) \quad y_t = y_t^x + E y_t = y_t^x + \mu_t.\]
Since the welfare function will accordingly be decomposed into the two parts involving $y^*_t$ and $\mu_t$ respectively, there is no cost in replacing (1.2) by the following slightly more general welfare function

$W = - \sum_{t=1}^{T} E y^*_t' K_{1t} y^*_t + \sum_{t=1}^{T} (- \mu^*_t K_{2t} \mu_t + 2k^*_t \mu_t)$,

which will reduce to (1.2) when the symmetric matrices $K_{1t}$ and $K_{2t}$ are equal. The econometric model (1.1), when the control equation (1.4) is applied, becomes

$y_t = (A + CG_t) y_{t-1} + Cg_t + b + u_t$.

The problem then is to maximize (1.8) with respect to $G_t$ and $g_t$, given the system (1.9). Section 2 will present a solution to the partial problem of maximizing the first term on the right of (1.8), while Section 3 will give a solution to the complete problem. Section 4 provides a solution by dynamic programming, for the purpose of relating the present work with the more standard works on the control of stochastic as well as non-stochastic systems. Section 5 contains a few concluding remarks.

For the readers who are not familiar with the previous paper [3], it should be pointed out that the problem here treated is more general than it might first appear. The econometric model may contain lagged variables of more periods, such as

$y_t = A_1 y_{t-1} + A_2 y_{t-2} + C_0 x_t + C_1 x_{t-1} + b + u_t$.
the control equation may actually be

\[ x_t = G_{1t}y_{t-1} + G_{2t}y_{t-2} + G_{3t}x_{t-1} + g_t; \]

the welfare function may involve both \( y_t \) and \( x_t \); and the residuals \( u_t \) may be serially correlated. By redefining a new vector \( y_t \) to include the previous \( y_t, y_{t-1} \), and \( x_t \) in the above example, we are able to convert the system into the form (1.1) and to deal only with \( y_t \) in the welfare function. In this conversion, we note that the number \( p \) of output variables \( y_t \) must be larger than the number \( q \) of control variables \( x_t \) which are imbedded in the former. Serial correlations in \( u_t \) can also be eliminated by suitable manipulations if \( u_t \) satisfies an autoregressive scheme with known parameters. The reader may wish to refer to the previous paper [3] for further motivation and for discussion of economic applications.

2. SOLUTION TO A PARTIAL PROBLEM

As in the previous paper [3], we first consider maximizing the welfare function

\[ W_1 = -\sum_{t=1}^{T} E y_t^* K_{lt} y_t^* = -\sum_{t=1}^{T} \text{tr} K_{lt} E y_t^* y_t^*. \]  

(2.1)

To obtain the model governing the deviation \( y_t^* \) from mean, assuming the control equation (1.4) is used, we take expectation of (1.9) to yield
\[ (2.2) \quad \mu_t = (A + CG_t)\mu_{t-1} + Cg_t + b \]

and subtract (2.2) from (1.9):

\[ (2.3) \quad y_t^* = (A + CG_t)y_{t-1}^* + u_t = R_t y_{t-1}^* + u_t \]

where \( R_t \) is defined as

\[ (2.4) \quad R_t = (A + CG_t) \]

Since the welfare function (2.1) involves the covariance matrices \(Ey_t'y_t^* = \Gamma_t\), say, and each \( \Gamma_t \) is a function of \( G_1, \ldots, G_t \) by (2.3), one may try to write \( W_1 \) as a function of \( G_1, \ldots, G_T \), and proceed to maximize. This direct approach turns out to be extremely difficult. The simple approach adopted, as in the previous paper [3], is to transform the problem to one of constrained maximization by treating, somewhat artificially, an additional set of variables \( \Gamma_1, \ldots, \Gamma_T \), and to consider the welfare function as a function of both sets of variables \( W_1 = W_1(\Gamma_1, \ldots, \Gamma_T; G_1, \ldots, G_T) \) while these variables are subject to constraints.

We will derive a set of constraint equations by first post-multiplying (2.3) by \( y_t' \) and taking expectation

\[ (2.5) \quad Ey_t'y_t^* = R_t E y_{t-1}^* y_t' + V \]

then by premultiplying the transpose of (2.3) by \( y_{t-1}^* \) and taking
(2.6) \( EY_t^* Y_{t-1}^* = (Ey_{t-1}^* Y_{t-1}^* )R_t' \).

Substitution of (2.6) into (2.5) gives the constraints,

(2.7) \( EY_t^* Y_t^* = V + R_t(Ey_{t-1}^* Y_{t-1}^* )R_t' \),

or

\[ \Gamma_t = V + (A + CG_t) \Gamma_{t-1} (A + CG_t)' \] (t=1,...,T). \]

Note that \( \Gamma_0 = 0 \), since \( Y_0 = \mu_0 \) and \( Y_0^* = 0 \), and that \( \Gamma_1 = V \).

The Lagrangian expression, with \( H_t \) denoting symmetric matrices of Lagrange multipliers, is

(2.8) \[ L_1 = -\sum_{t=1}^{T} tr K_t \Gamma_t + \sum_{t=1}^{T} tr H_t[\Gamma_t - (A + CG_t) \Gamma_{t-1}(A + CG_t)'] - V \].

We differentiate \( L_1 \) with respect to \( G_t \) and \( \Gamma_t \) (t=2,...,T), using the differentiation rule

(2.9) \[ \frac{\partial}{\partial G} tr BG = \frac{\partial}{\partial G} tr GB = B' \],

yielding

(2.10) \( \frac{\partial L_1}{\partial G_t} = -2C'H_tA\Gamma_{t-1} - 2C'H_tC CG_t \Gamma_{t-1} = 0 \) (t=2,...,T);

(2.11a) \( \frac{\partial L_1}{\partial \Gamma_t} = -K_t + H_t - (A + CG_{t+1})' (H_{t+1}(A + CG_{t+1}) = 0 \) (t=2,...,T-1);

(2.11b) \( \frac{\partial L_1}{\partial \Gamma_T} = -K_T + H_T = 0 \).

Equations (2.10), (2.11) and (2.7) will determine the unknowns \( G_t, H_t, \) and \( \Gamma_t \).
To solve for $H_t$ and $G_t$, we simply use (2.11) and (2.10) and work backwards. Given $H_T = K_1T$ from (2.11b), we set, from (2.10),

$$G_T = -(C'H_T C)^{-1} C'H_T A.$$  

Then from (2.11a), we obtain

$$H_{T-1} = K_{1,T-1} + (A + CG_T)' H_T (A + CG_T).$$

The process continues until $H_2$ and $G_2$ are obtained. As we have noted in connection with equation (2.7), $r_0 = 0$ and $r_1 = V$. Hence $G_1$ has no effect on $r_1$, and we may as well set $G_1 = 0$. This does not mean that nothing is done to control the economy for the first period; any control can be exercised through the choice of $g_1$ in the control equation (1.4).

In the previous paper [3], we dealt with the steady-state solution, with the control matrix $G$ invariant through time after equilibrium is reached. The optimal $G$ was obtained by solving the two equations

$$G = -(C'HC)^{-1} C'HA$$

and

$$H = K_1 + (A + CG)' H (A + CG).$$

These equations will give the steady-state solution to the above matrix difference equations in $G_t$ and $H_t$, if such a steady state exists for sufficiently large $T$ and for $K_{1t} = K_1$ for all $t$. We also note that, in the present formulation, the matrices $A$ and $C$ can be functions of $t$; the subscript $t$ can be added to $A$ and $C$ in the appropriate places in equations (2.8), (2.10) and (2.11).
As it has been pointed out in the previous paper [3], if there are delays in obtaining information and/or in carrying out decisions, the control matrix $G_t$ can be partitioned into

$$(2.14) \quad G_t = \begin{pmatrix} G_{1t} & G_{2t} \end{pmatrix} = \begin{pmatrix} 0 & G_{2t} \end{pmatrix},$$

$G_{1t} = 0$ being coefficients of the components of $Y_{t-1}$ in the control equation (1.4) for which no information is available at time $t$; the covariance matrix $\Gamma_t$ can be accordingly partitioned

$$(2.15) \quad \Gamma_t = \begin{pmatrix} \Gamma_{11,t} & \Gamma_{12,t} \\ \Gamma_{21,t} & \Gamma_{22,t} \end{pmatrix},$$

and equation (2.10) will be replaced by

$$(2.16) \quad \frac{\partial L_1}{\partial G_{2t}} = -2C'H_tA\Gamma_{2,t-1} - 2C'H_tC\Gamma_{22,t-1} = 0,$$

but equations (2.11) and (2.7) will remain the same.

To solve equations (2.16), (2.11) and (2.7) for $G_{2t}$, $H_t$ and $\Gamma_t$ ($t=2,\ldots,T$), the following iterative method is suggested. Start with an initial guess $G^{(1)}_{2,t}$ ($t=2,\ldots,T$) in the first iteration. For example, these matrices could be the appropriate columns selected from the solution for $G_t$ ($t=2,\ldots,T$) obtained by equations (2.10) and (2.11) when there are no information delays. Using these
$G^{(1)}_{2,t}$, calculate $H^{(1)}_t (t=T,T-1,\ldots,2)$ by equation (2.11), and $\Gamma^{(1)}_t (t=2,3,\ldots,T)$ by equation (2.7). In the second iteration, compute $G^{(2)}_{2,t}$ by equation (2.16), i.e.,

$$G^{(2)}_{2,t} = -(C'H^{(1)}_t C)^{-1} \Sigma H^{(1)}_t A\Gamma^{(1)}_{t-1} \Gamma^{(1)}_{2,t-1}^{-1} \Gamma^{(1)}_{2,t-1} \Gamma^{(1)}_{2,t-2} \Gamma^{(1)}_{2,t-2} (t=2,\ldots,T).$$

These matrices will serve as inputs for calculating $H^{(2)}_t$ by equation (2.11) and $\Gamma^{(2)}_t$ by equation (2.7), and so forth.

3. SOLUTION TO THE COMPLETE PROBLEM

Return now to the maximization of the original welfare function

$$W = \sum_{t=1}^{T} \text{tr} K_{t} \Gamma_{t} - \sum_{t=1}^{T} \mu_{t} K_{2,t} \mu_{t} + \sum_{t=1}^{T} 2k_{t} \mu_{t}.$$  

Since we have the constraints (2.2) and (2.7) respectively for $\mu_{t}$ and $\Gamma_{t}$, one possible approach is to form the Lagrange expression, using additional vectors $\lambda_{t}$ of multipliers,

$$L = \frac{1}{2} W + \frac{1}{2} \sum_{t=1}^{T} \text{tr} H_{t} [\Gamma_{t} - (A + CG_{t}) \Gamma_{t-1} (A + CG_{t})'] - V$$

$$- \sum_{t=1}^{T} \lambda_{t}[\mu_{t} - (A + CG_{t}) \mu_{t-1} - CG_{t} - b].$$

Straightforward differentiation yields

$$\frac{\partial L}{\partial G_{t}} = \frac{1}{2} \frac{\partial L}{\partial G_{t}} + C' \lambda_{t} \mu_{t-1} = 0 \quad (t=2,\ldots,T)$$
\((3.4) \quad \frac{\partial L}{\partial \Gamma_t} = \frac{1}{2} \frac{\partial L_1}{\partial \Gamma_t} = 0 \quad (t=2, \ldots, T)\)

\((3.5) \quad \frac{\partial L}{\partial g_t} = c'\lambda_t = 0 \quad (t=1, \ldots, T)\)

\((3.6a) \quad \frac{\partial L}{\partial \mu_t} = K_{2t}\mu_t + k_t - \lambda_t + (A + CG_{t+1})'\lambda_{t+1} = 0 \quad (t=1, \ldots, T-1)\)

\((3.6b) \quad \frac{\partial L}{\partial \mu_T} = K_{2T}\mu_T + k_T - \lambda_T = 0 .\)

Using equation \((3.5)\), we find that \((3.3)\) and \((3.4)\) are equivalent to \((2.10)\) and \((2.11)\) for the solution of optimal \(G_t\) and \(B_t\) in section 2. Thus policies to control the covariances are still obtained and by the previous method are unaffected by the need for controlling the mean \(\mu_t\). This point can be observed directly from equation \((2.2)\).

Suppose that \(G_t\) maximize \(W_1\), and that \(G_t^*\), together with \(g_t^*\), say, maximize the remaining part of \(W\), i.e.,

\((3.7) \quad W_2 = - \sum_{t} \mu_t K_{2t} \mu_t + \sum_{t} 2k' \mu_t .\)

But \(G_t\) can always be used to generate the same \(\mu_t\) as \(G_t^*\) can, since, by \((2.2)\),

\((3.8) \quad \mu_t = (A + CG_t^* )\mu_{t-1} + CG_t^* + b \)

\[= (A + CG_t)\mu_{t-1} + C[G_t^* - G_t] \mu_{t-1} + g_t^* + b .\]

All that needs to be done is to replace \(g_t^*\) by \(g_t = (G_t^* - G_t) \mu_{t-1} + g_t^*\). This also shows that the maximum of \(W_2\) can be achieved for any arbitrary set of \(G_t\).
To obtain the optimal \( g_t \), we could solve equations (3.5), (3.6) and (2.2) for \( g_t, \mu_t \) and \( \lambda_t \) \((t=1, \ldots, T)\), assuming that the optimal \( G_t \) have already been found by the method of section 2. These are \( T \times (q+2p) \) linear equations in the unknowns \( g_1, \ldots, g_T, \mu_1, \ldots, \mu_T, \lambda_1, \lambda_T \). This problem is an example of a non-stochastic control problem in discrete time where the objective function \( W_2 \) is quadratic and the dynamic model (2.2) is a system of linear difference equations in the state variables \( \mu_t \) and the control variables \( g_t \). This is precisely the control problem when our model (1.1) becomes non-stochastic, with \( \mu_t = 0 \). We have just pointed out that maximum of welfare in this case can be achieved for any set of \( G_t \), including \( G_t = 0 \). This means that, in the world of certainty, optimal \( x_t \) can be set independently of recent information \( y_{t-1} \), and no feedback relation will be required.

Methods for solving this elementary non-stochastic control problem have been treated extensively in the literature, e.g. [2, chapter 6], and will not be repeated here. However, since this non-stochastic control problem is imbedded in the larger stochastic control problem, one can utilize the results on \( G_t \) and \( H_t \) to obtain an easier method of solution than merely solving a large set of simultaneous linear equations (3.5), (3.6) and (2.2). In particular, if we let \( K_{1t} = K_{2t} = K_t \) in the welfare function, and apply the relationship (2.11) between \( H_t, G_t, \) and \( K_t \), we can rewrite (3.6) as
\[(3.9) \quad \lambda_t = k_t - K_t \mu_t + R_{t+1}^t \lambda_{t+1} \]

\[= k_t - H_t \mu_t + R_{t+1}^t H_{t+1} R_{t+1}^t \mu_t + R_{t+1}^t \lambda_{t+1} \quad .\]

By the use of (2.2) to replace \( R_{t+1}^t \mu_t \), and (2.10) to nullify \( C'HR_t \), (3.9) becomes

\[(3.10) \quad (\lambda_t + H_t \mu_t) = k_t - R_{t+1}^t [H_{t+1} b - (\lambda_{t+1} + H_{t+1} \mu_{t+1})] \]

or

\[h_t = k_t - R_{t+1}^t (H_{t+1} b - h_{t+1}) \quad ,\]

for the newly defined \( h_t \). If we premultiply equation (2.2) by \( C' H_t \) and solve for \( g_t \), using also equations (3.10), (3.5) and (2.10), we will obtain

\[(3.11) \quad g_t = -(C' H_t C)^{-1} C' (H_t b - h_t) \quad .\]

Equations (3.10) and (3.11) can very easily be solved. The former is a linear difference equation in \( h_t \) and can be solved backward in time, from \( h_T = k_T \). Given \( h_t \), the optimal \( g_t \) will then be obtained by (3.11).

4. SOLUTION BY DYNAMIC PROGRAMMING

Recently, there has been a growing interest among economists in optimal control theory for non-stochastic systems, as exemplified by the excellent expository article of Dorfman [4]. It may be useful to provide an elementary exposition of some aspects of optimal control theory for stochastic systems as it is treated in the literature,
e.g., [7], [8], and [9], and to relate it to control theory for non-stochastic systems.

In this paper, we have treated a stochastic control problem in section 2, and a non-stochastic control problem in section 3. In the former, the dynamic model (2.3) is stochastic, and the welfare function (2.1) is in the form of a mathematical expectation. In the latter, the dynamic model (2.2), with \( \mu_t \) treated as output variables and \( q_t \) as control variables, is non-stochastic or deterministic. These two problems should serve to illustrate the nature of stochastic and non-stochastic control problems. The method that we have used to solve both problems is the method of Lagrange multipliers.

In this section, we will solve the problem of maximizing the welfare function (1.2) by dynamic programming [1], which is a method commonly used to solve stochastic as well as non-stochastic control problems [4], [7], [8], and [9]. Readers familiar with Dorfman's paper [4] will see that the same principle of optimality is being applied here to a problem of stochastic control.

Let us first restate the problem. Given a dynamic stochastic model in discrete time

\[
(4.1) \quad Y_t = AY_{t-1} + CX_t + b + u_t,
\]

and a welfare function of the form

\[
(4.2) \quad W(s, y_s; x_{s+1}, \ldots, x_T) = E \sum_{t=s}^{T} (-y'K_y Y_t + 2k_t'y_t),
\]

find \( x_{s+1}, \ldots, x_T \) to maximize \( W \), with each \( x_t \) to be determined by the linear feedback control equation
\[(4.3) \quad x_t = G_t y_{t-1} + g_t .\]

Note that welfare depends on the initial period \( s \), the initial value \( y_s \) of the state variable in that period, and on the values of future control variables \( x_{s+1}, \ldots, x_T \). The expectation in \((4.2)\) is conditional on \( y_s \).

Let \( W^*(s, y_s) \) denote the maximum welfare obtainable given the initial value \( y_s \) in period \( s \), with \( x_t \) determined by the optimal control parameters \( G_t^* \) and \( g_t^* \) \((t = s+1, \ldots, T)\) according to the equation \((4.3)\). It will first be observed that \( W^* \) must take the form

\[(4.4) \quad W^*(s, y_s) = -y_s' H_s y_s + 2h_s' y_s - p_s \]

for some symmetric matrix \( H_s \), some vector \( h_s \), and some scalar \( p_s \). To see this, we note that each \( y_t \) can be written as

\[(4.5) \quad y_t = (A + C G_t^*) y_{t-1} + C g_t^* + b + u_t \]

\[= R_t y_{t-1} + C g_t^* + b + u_t \]

\[= u_t + R_t u_{t-1} + R_t R_{t-1} u_{t-2} + \ldots + (R_t R_{t-1} \ldots R_{s+2}) u_{s+1} \]

\[+ (C g_t^* + b) + R_t (C g_{t-1}^* + b) + \ldots + (R_t R_{t-1} \ldots R_{s+2}) (C g_{s+1}^* + b) \]

\[+ (R_t R_{t-1} \ldots R_{s+1}) y_s , \]

that the expectation of \(-y_t' K_t y_t + 2k_t' y_t\) must be quadratic in \( y_s \), and that the welfare function \((4.2)\) is a sum of these quadratic functions, thus having the form \((4.4)\).
Once the form of $W^*(s,y_s)$ is ascertained, we can apply the principle of optimality in dynamic programming:

$$W^*(s,y_s) = \max_{x_{s+1}} E\{-y'_s K_s y_s + 2k'_s y_s + W^*(s+1,y_{s+1})\}.$$  

According to this equation, the maximum welfare from period $s$ on is the maximum, with respect to $x_{s+1}$, of the sum of the welfare contribution of this period and the maximum of welfare from the next period on. The expectation in (4.6) is conditional on $y_s$; the term $W^*(s+1,y_{s+1})$ depends on $y_{s+1}$ which is yet unknown, or is a random vector, from the vintage point of period $s$. Using the form (4.4) for $w^*$, we can write the term to be maximized as

$$(4.7) \quad -y'_s K_s y_s + 2k'_s y_s + E\{-y'_{s+1} H_{s+1} y_{s+1} + 2h'_{s+1} y_{s+1} - p_{s+1}\}$$  

$$= -y'_s K_s y_s + 2k'_s y_s - (Ay_s + Cx_{s+1} + b)' H_{s+1} (Ay_s + Cx_{s+1} + b)$$  

$$- E u'_{s+1} H_{s+1} u_{s+1} + 2h'_{s+1} (Ay_s + Cx_{s+1} + b) - p_{s+1}.$$  

Maximizing (4.7) with respect to $x_{s+1}$ by differentiation yields

$$(4.8) \quad -2C'H_{s+1}(Ay_s + Cx_{s+1} + b) + 2C'h_{s+1} = 0;$$  

$$(4.9) \quad x_{s+1} = G_{s+1} y_s + g_{s+1},$$  

where

$$(4.10) \quad G_{s+1} = - (C'H_{s+1} C)^{-1} C' H_{s+1} A ,$$  

$$(4.11) \quad g_{s+1} = - (C'H_{s+1} C)^{-1} C' (H_{s+1} b - h_{s+1}).$$
Thus, the optimal control parameters are given by (4.10) and (4.11). To find $H_S$ and $h_S$, we substitute (4.9) back to (4.7), and equate the result to the left-hand side of (4.6). The maximum of (4.7) is

\[(4.12)\quad - y_s'K_s y_s + 2k_s'y_s - y_s'(A + CG_{s+1})'H_{s+1}(A + CG_{s+1})y_s \]

\[- (CG_{s+1} + b)'H_{s+1}(CG_{s+1} + b) - 2(CG_{s+1} + b)'H_{s+1}(A + CG_{s+1})y_s \]

\[- \text{tr} H_{s+1}V + 2h_{s+1}'(A + CG_{s+1})y_s + 2h_{s+1}'(CG_{s+1} + b) - p_{s+1} \]

which should be equated to

\[(4.13)\quad W^s(s, y_S) = y_s'H_S y_s + 2h_s'y_s + p_s, \]

yielding the following two difference equations for $H_S$ and $h_S$:

\[(4.14)\quad H_S = K_S + (A + CG_{s+1})'H_{s+1}(A + CG_{s+1}), \]

\[(4.15)\quad h_S = k_S - (A + CG_{s+1})'H_{s+1}(CG_{s+1} + b) + (A + CG_{s+1})'h_{s+1} \]

\[= k_S - (A + CG_{s+1})'(H_{s+1}b - h_{s+1}). \]

Equations (4.10) and (4.14) are identical with equations (2.10) and (2.11) of section 2; they can be used to determine $G_S$ and $H_S$. Given $G_S$ and $H_S$, equations (4.11) and (4.15) can be used to determine $g_S$ and $h_S$. They are identical with equations (3.11) and (3.10) respectively.

The exposition of this section has, in fact, also demonstrated the solution of a non-stochastic control problem by dynamic programming, since we could easily convert the stochastic model (4.1)
into a non-stochastic model by letting $u_t = 0$; the welfare function (4.2) could remain unchanged since the expectation $E$, though unnecessary in the non-stochastic case, is harmless. We could follow the same development from equation (4.2) on, noting $u_t = 0$ in all places, thus $V = 0$ in (4.12), and still arriving at the same results (4.10), (4.11), (4.14) and (4.15). This exposition is presented because it ties together the theories of stochastic and non-stochastic control, and because, to the author's knowledge, an elementary treatment of stochastic control theory using dynamic programming is not available in the literature.

5. CONCLUDING REMARKS

In this paper, we have replaced the assumptions of a previous paper [3] that the welfare function is constant through time and that the linear econometric system under control is stationary, by the assumptions that the welfare function is a sum for a finite number of periods and that the econometric system under control may be non-stationary. We still believe that the assumptions of the previous paper [3] are relevant for many economic applications, especially when the variables $Y_t$ are interpreted as the first differences of certain economic variables, or perhaps of their logarithms. Under these assumptions, the present paper provides the transient solution to supplement the steady-state solution given earlier.

In some economic applications, one may wish to relax these assumptions. For example, if one wishes to have the variables $Y_t$ (first differences or not) follow certain target vector $a_t$, in the
sense of minimizing the sum, over $t$, of certain positive semi-
definite quadratic forms in the deviations $y_t - a_t$, the welfare
function can be reformulated as in this paper, namely

$$W = E \left[ - \sum_{t=1}^{T} (y_t - a_t)' K_t (y_t - a_t) \right]$$

$$= E \left[ - \sum_{t=1}^{T} (-y_t K_t y_t + 2a_t' K_t y_t) \right] - \sum_{t=1}^{T} a_t' K_t a_t$$

which is equivalent to (1.2). There is also the possibility that
the system under control may be non-stationary. Under these circum-
stances, the solution given in this paper will be applicable.

However, even if the objective is to follow closely certain
growing target vector $a_t$, the assumptions of the previous paper [3]
might still be valid. As an example, let $a_t$ satisfy the difference
equation $a_t = Da_{t-1}$, where $D$ is a diagonal matrix with some
diagonal elements greater than one. We may augment system (1.1) by
the above difference equation, redefine a new output vector to include
both $y_t$ and $a_t$, and consider maximizing the welfare function

$$W = E \left[ - \sum_{t=1}^{T} (y_t - a_t)' K (y_t - a_t) \right]$$

$$= E \left[ - \sum_{t=1}^{T} (y_t' a_t') \begin{pmatrix} K & -K \\ -K & K \end{pmatrix} \begin{pmatrix} y_t \\ a_t \end{pmatrix} \right]$$

The parameters of this welfare function are time-invariant. Equations
(2.10) and (2.11) might have a stationary solution for $G$ and $H$,
identical with the solution of the previous paper [3].
While relaxing the assumptions of the previous paper, we have applied the same method, that of Lagrangian multipliers, to solve the problem of this paper. We have also provided an exposition of the method of dynamic programming to solve a stochastic and a non-stochastic control problem, thus pointing out a unifying element in optimal control theory for stochastic and non-stochastic systems. It is hoped that the methods of these two papers will be applied to empirically relevant econometric models and welfare functions, so as to contribute to our knowledge of optimal policies for economic stability and growth.
REFERENCES


FOOTNOTES

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1. In the original model of Simon [5], the control variable, pro-
duction, was assumed to depend on past sales, and sales were the
random variables in the model. Simon's formulation was quite
natural, but when Theil [6] adopted Simon's model to an econometric
system, the variables on which $x_t$ depends became the somewhat un-
natural $s_1, \ldots, s_{t-1}$ and $E_{s_t}, \ldots, E_{s_T}$ in equation (1.3).

2. Of course, the inverse of $C'H_T'C$ might not exist. In this case,
the optimal reaction coefficients $G_T$ are not unique, and we
interpret $(C'H_T'C)^{-1}$ as a generalized inverse.

3. One fairly obvious approach is to reduce the size of this linear
problem by eliminating the variables $\mu_1, \ldots, \mu_T$ and $\lambda_1, \ldots, \lambda_T$.
Using the model (2.2), each $\mu_t$ becomes a linear function of
$g_1, \ldots, g_t$, and $\mu_1, \ldots, \mu_T$ become linear functions of $g_1, \ldots, g_T$.
The welfare function (3.7) then becomes a quadratic function in
$g_1, \ldots, g_T$, and can be easily maximized with respect to these
remaining variables.