SPECTRAL ANALYSIS OF NEW YORK STOCK MARKET PRICES

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Summary

New York stock price series are analyzed by a new statistical technique. It is found that short-run movements of the series obey the simple random walk hypothesis proposed by earlier writers, but that the long-run components are of greater importance than suggested by this hypothesis. The seasonal variation and the "business-cycle" components are shown to be of little or no importance and a surprisingly small connection was found between the amount of stocks sold and the stock price series.

1. The Random Walk Hypothesis

The stock market is an institution of considerable interest to the public at large and of real importance to students of a nation's economy. The variables which make up a stock market may not directly affect the mechanism of the economy but they certainly influence the psychological climate within which the economy works. To the extent to which the movements of the economy directly affect the stock market, a feedback situation occurs, although there are reasons to suspect that the strength of the feedback is not strong. The stock market produces large amounts of high quality data derived from well-understood variables. Despite these facts, the stock market has attracted surprisingly little study by professional economists or statisticians. It is interesting to note that Carl Menger is reported to have made careful studies of the Vienna stock market before,

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in 1871, developing the concept of marginal utility and his theory of prices.
Later, the French mathematician, M. L. Bachelier, published his Théorie de la
Spéculation (1900) in which the random character of stock price movements was
derived. While strictly theoretical studies are rare, the stock market having
been largely neglected in the development of price theory, there exist, of
course, descriptive works, too numerous to mention. Of theoretical-statistical
works, first reference has to be made to the studies by A. Cowles\(^1\). In the
last few years, however, a number of studies have appeared: Kendall\(^2\), Alexander\(^3\),
Cootner\(^4\), Osborn\(^5\), etc., chiefly concentrating on the weekly price movements
either of particular stocks or of the over-all market.

In this paper we restrict ourselves as do most of the above-mentioned
authors to prices of the New York Stock Exchange and, strictly speaking, our
results apply only to this sample. There is, however, little likelihood that
stock markets in other countries will behave very differently. Nevertheless
studies of other markets are indicated. The various authors though, starting


Cowles, A. and Jones, H.E., "A Revision of Previous Conclusions Regarding


3) Alexander, S., "Price Movements in Speculative Markets: Trends or Random

4) Cootner, P. H., "Stock Prices: Random vs. Systematic Changes", Industrial

5) Osborn, M. F. M., "Brownian Motion in the Stock Market", Operations Research,

Osborn, M. F. M., "Periodic Structure in the Brownian Motion of Stock Prices",
from different viewpoints and developing slightly different hypotheses, have
been surprisingly consistent in their general conclusions, which are that these
price series can be represented by a very simple model: If $x_t$ is the (discrete)
price series the model suggested is that the first differences of this series
appear to be purely random; in symbols

$$x_t - x_{t-1} = \epsilon_t$$

where $\epsilon_t$ has mean zero and is uncorrelated with $\epsilon_{t-k}$, all $k \neq 0.1)$ (Henceforth,
series such as $\epsilon^t$ will be called 'white noise'.). This model has been called a
"random walk" model and, although this designation is not entirely correct as
steps of an equal length are allowable this term will be used to describe the
model throughout this paper. Thus, if the model is correct, the series will proceed
by a sequence of unconnected steps, starting each time from the previous value of
the series.

The phenomenon of a random walk of prices will arise if a large number
of people are continually predicting price and use as their best predictive model
that the price at the next moment of time is the same as the present value,
the time units between readings being minutes, hours, days or even possibly weeks.
The model based on this observation appears to fit the data extremely well and
Cootner properly finishes his paper with this remark: "If their (stock price
series) behavior is more complicated than the random walk model suggests, it will
take more sophisticated statistical testing to discover it." It is one of the
objects of the present paper to suggest such a sophisticated method. We shall
show that, although the random walk model does appear to fit the data very well,
there are certain prominent long-run features of the variables which are not
explained by this model.

1) In some of the above-mentioned papers, the series $x_t$ was transformed first,
e.g. Osborn considers $\log x_t$. The need for such transformations is dis-
cussed in Appendix A.
Further objects of the present paper are to promote the idea that stock market data (and particularly stock exchange 'folk-lore') should be investigated by rigorous methods and that the most appropriate statistical techniques to be used in such an investigation are the recently-developed spectral methods. These have already been used with considerable success in other fields of research and, although they have required considerable adaptation and improvement before being entirely applicable to economic series we contend that spectral analysis has now reached a stage of development where it can be used with some confidence on economic series. A group of workers at the Econometrics Research Program of Princeton University has been studying these methods with particular reference to economics and the results so far achieved are shortly to be published in book form\textsuperscript{1}). Only a few of the manifold methods are used in this paper.

The series analyzed by us for this investigation were various Securities and Exchange Commission's (S.E.C.) weekly price series (1939-1961), monthly price series for six U. S. companies (1946-1960), weekly price series together with the weekly turnover of their stocks for two further companies (1957-1961), the Standard and Poor common stock price index (monthly, 1915-1961); Dow-Jones Index 1915-1961. (Full details may be found in Appendix B.)

In the next section, a brief, non-mathematical description of the ideas underlying spectral methods is given and this is followed by an account of the more important results achieved by these methods when applied to data drawn from the New York Stock Exchange. The various technical problems that arise during this work are described in Appendix A and a full list of the series so far analyzed together with the major results are given in Appendix B.

\textsuperscript{1)} Analysis of Economic Time Series by C. W. J. Granger, in association with M. Hatanaka, Princeton University Press (1963). Particular chapters of this reference will henceforth be denoted by Granger \textsuperscript{op. cit.} \textsuperscript{Chapter 9}, etc. The general philosophy underlying these studies is described in C. Morgenstern: "A New Look at Economic Time Series Analysis", in Money, Growth and Methodology, in honor of J. Akerman, H. Hegeland (Edit.) Lund 1961, pp. 261-272.
2. **Spectral Methods**

Consider a discrete time series having neither trend in mean, trend in variance or trend in correlation between the present value and any past value. Such a series, which we will call stationary, essentially has the property that the underlying laws which generate it are invariant over time. Since A. Schuster's work of the beginning of this century, it has been realized that a finite stationary series can be represented by a finite number of Fourier terms and an infinite series can be represented by a countably infinite number of such terms. Only in the last twenty years has it been suggested that results, both more general and more rigorously established, may be achieved by using a more general representation.

It may be useful to illustrate this idea by using an analogy between an ordinary statistical variable (such as height of a given population) and the Fourier representation of a stationary series\(^1\). A term which first needs to be defined is "frequency" which is proportional to the inverse of the well understood concept of 'period'. Thus, if a component has a period of \(k\) weeks, when weekly data are being used, this component will have the frequency \(2\pi/k\).

Just as it is not generally assumed that a statistical variable has only a finite number of possible values (as would a variable with a binomial distribution) or a countably infinite number of possible values (as would a variable with Poisson distribution), there is no good reason why a series need be represented by a finite or a countably infinite number of frequencies in the Fourier representation. Greater generality is achieved by allowing the statistical variable to take all possible values (as is the case when a variable

\(^1\) It is hoped that this analogy is not confusing; the type of statistical variable envisaged is an independent sample and the fact that the time series is itself a statistical variable is not here taken into account.
is normally distributed). Similarly we achieve greater generality by using all frequencies in the Fourier representation of the series. The extra degree of generality involved is exactly that of moving from an infinite series to an integral over the whole real line.

When considering a statistical variable we are usually interested in the probability of some event. If the variable $X$ is distributed normally, then the probability that $X$ equals some given constant is zero, but the probability that it lies in some given interval is non-zero. By suitably choosing the intervals we are able to define the useful frequency function $f(x)$ by using

$$f(x) \, dx = \text{Prob} \left( x \leq X \leq x + dx \right).$$

In practice, we are unable to estimate $f(x)$ for all $x$ as only a finite amount of data is available and so we use the histogram as an estimate.

When considering the Fourier representation of a time series we are most interested in the contribution that some particular frequency component makes to the over-all variance. In the most general representation of the series (using a Fourier integral rather than a Fourier series) no one frequency contributes any finite amount to the variance but the set of all frequencies lying in some interval will contribute a positive amount. If the frequency band $(\omega, \omega + d\omega)$ contributes $f(\omega)d\omega$ to the total variance, we call the function $f(\omega)$ the power spectrum of the series. Just as in the case of the frequency function, we are unable to estimate $f(\omega)$ for every $\omega$ and therefore we have to form estimates of $f(\omega)$ over a set of frequency bands. A large value of estimated $f(\omega)$ indicates that the frequencies of this band are of particular importance as they contribute a large amount of the total variance.
Figure 1 shows the logarithm of the estimated spectrum (at 120 frequency points) of the monthly New York commercial paper rate series for the period 1876-1914. This series was chosen as an illustration as it was found to contain cycles of a conventional type (40 months) and an important but not overpowering annual (seasonal) component. We note that the bands centered on components with periods of 40 and 12 months are of particular importance, together with the harmonics of the annual component. The first of these bands contributes 40 percent of the variance, the annual component contributes 17 percent. (cf. Fig. 1)

The spectrum of an independent sequence of terms (i.e. white noise) is a horizontal line parallel to the x-axis over the complete frequency range $(0, \pi)$. But economic series frequently have spectra of this shape:

![Fig. 2](image)

Such a spectral shape can be interpreted as implying that the longer the period of the component the larger is its amplitude. This is not particularly surprising as in the past many cycles have been seen by economists in their data and if such 'cycles' had not been of obvious importance they would have been indistinguishable from the ever-present background noise.

1) This important short-term interest rate series is fully discussed in Oskar Morgenstern: International Financial Transactions and Business Cycles, Princeton, 1959. The 90% confidence bonds are of constant width about the estimated spectrum. Their size is shown at zero frequency.
The section of the power spectrum of most interest to economists is the low-frequency range, within which all the 'long-run' components are concentrated. Unfortunately, this range is also the most difficult to deal with. Any trend in mean in the series will give the zero frequency band a large value and, due to the estimation methods used, this large value will be inclined also to raise the values of the neighboring frequency bands. (This effect is known as 'leakage'.) Thus, before the low frequencies can be studied, a trend in mean would have to be removed. Various methods are available (see Granger, op. cit. Chapter 8) but a long, equal-weight moving average of the series subtracted from the series is both convenient and easily appreciated. A second problem connected with the low-frequency bands is the fact that one requires several repetitions of a periodic component before its amplitude can be estimated at all accurately.

Interpretation of the terms 'low-frequency' or 'long-run' depend both upon the length of the available data and the time-interest used. A component that is extremely long-run in a series of 700 pieces of weekly data is only moderately so in a series of monthly data of comparable length. It is for this reason that we can say nothing about the business-cycle component of the Securities and Exchange Commission's weekly price series even though we used data for more than a thousand weeks, but in our analysis of the monthly Standard and Poor price series, which was shorter but covered a longer period of time, information about this component could be extracted. Oscillations with periods comparable in length with the length of series available will then be grouped with 'trend' when spectral methods are used.
The existence of a trend in mean is only one way in which a time-series can show its non-stationary character and is the easiest of all to deal with. Trends in variance or in underlying structure also invariably occur in economics and these cannot be entirely removed. As spectral methods assume series to be stationary and as virtually no economic series are stationary it would seem that spectral methods are not appropriate for use in economics. However, a number of studies at Princeton, both theoretical and experimental, indicate that as long as the underlying structure of the series is not changing quickly with time, spectral analysis may be used with confidence. (This work is described in Granger, op. cit. Chapter 9 and methods of analyzing non-stationary series are introduced in Granger, op. cit. Chapter 10.)

This result also holds true when cross-spectral methods are used. Whereas a power-spectrum, by an extension of the classical analysis of variance, analyzes a single series, cross-spectral methods investigate the relationship between a pair of series.

It has been explained above that a series may be decomposed into a (possibly infinite) number of frequency bands. Let us call the \( j^{th} \) band \( \lambda_{jt} \), so that the sum of these terms over \( j \) will be the original series. As the time series is itself a statistical variable, so all the \( \lambda_{jt} \)'s will be statistical variables and it is an interesting property of a stationary series that \( \lambda_{jt} \) and \( \lambda_{kt} \) are uncorrelated, where \( k \neq j \).

If now we consider two series \( X_t, Y_t \) and decompose each into corresponding components \( \lambda_{jt}(x) \) and \( \lambda_{jt}(y) \) then, if both series are stationary, \( \lambda_{jt}(k) \) and \( \lambda_{kt}(y) \) will be uncorrelated. Thus, the two series are connected only via the correlation between the corresponding frequency bands \( \lambda_{jt}(x) \) and \( \lambda_{jt}(y) \). These correlations need not be the same for all pairs of frequencies,
there being no reason why two long-run components need be as highly correlated as two short-run components of a given pair of time series. The diagram which plots the square of these correlations against frequency is known as the coherence-diagram. (Fig. 6 shows such diagrams.) Similarly, it is possible that one of any pair of corresponding frequency bands will be time lagged to the other, but again this time-lag need not be the same for all pairs of frequencies. Indeed, it seems likely that if one series is inherently lagged to the other, then the long-run components will have a longer lag than do the short-run components. Such lags may be estimated directly from a diagram known as the phase-diagram.

It should be clear that use of such methods allow very complex relationships to be discovered and interpreted in a useful way.

An example of such a relationship was found when the cross-spectra were estimated between the monthly New York call money rate and the New York commercial paper rate for the period 1876 to 1914. It was found that all components of the former were leading the corresponding components of the latter, but that the lead was proportional to the period for each pair. Thus, the long-run components of the call money rate were leading the long-run components of the commercial paper rate by an amount that was larger than the corresponding lead for the short-run components.

Spectral methods are not necessarily easy to use, but we shall illustrate their power and value in the next section by discussing results on stock market data.
3. Results of the Analysis

Our first consideration was to test if the random walk model appeared to fit the price data without any transformations. We ignored any trend in variance (see Appendix A) and estimated the spectrum of the first difference of various of the major indices of the Securities and Exchange Commission. In every case the resulting spectrum was very flat over the whole frequency range, apart from a few exceptional frequencies which will be discussed later. Fig. 3 shows a typical resulting spectrum, that of the first difference of the S.E.C. composite weekly stock price index for the period January 7, 1939 to September 29, 1961. The initial result was that the simple random walk model fits the data very well as very few points are outside the 95% confidence limits.

However, there are various important implications about series generated by such a model. The theoretically significant one that the resulting variance is infinite need not concern us in practice, as explained in Appendix A. The fact that the simple random walk model tells us little about the long-run (low frequency) properties of the series is more important. This is easily seen by noting that if $X_t$ is a series obeying the random walk model and $Y_t$ is a second series defined by

$$ Y_t = X_t + a \cos \omega t $$

then we have

$$ X_t - X_{t-1} = \epsilon_t $$

and

$$ Y_t - Y_{t-1} = b \cos (\omega t + \theta) + \epsilon_t, $$
where

\[ b^2 = 2a^2 (1 - \cos \omega) \]

and

\[ \tan \theta = \frac{\sin \omega}{1 - \cos \omega} \]

Thus, if \( \omega \) is small, \( b \) will be small and the first differences of the two series \( X_t, Y_t \) are virtually indistinguishable although the latter contains a (possibly) important long-run component. As an example, with weekly data if the frequency \( \omega \) corresponds to periods of 28 weeks or 62 weeks, then the ratio of \( b^2 \) to \( a^2 \) will be 5% and 1% respectively.

To test whether the original stock price series contain long-run components of greater importance than the random walk hypothesis implies, the spectra for the original series were estimated after trend had been removed by the use of two long simple moving averages of lengths 104 and 78 weeks. It is necessary to remove the trend in mean first in order to ensure that no 'leakage of power' has occurred from the zero frequency band to neighboring bands. The effect on the spectrum of applying such moving averages is to multiply it by a known function (described in Appendix A). When the spectrum is estimated at 156 points, the function's value is zero at the first point, it reduces the size of the next three points, but the remaining frequency bands are not appreciably affected.

Figure 4 shows the estimated spectrum of the weekly S.E.C. composite price index for the same period as before. Superimposed upon the estimated spectrum is the expected spectrum if the random walk hypothesis were true, taking into account the effect of applying the moving averages, together with the 90% 'confidence bands' appropriate to this hypothesis. The insert in Fig. 4 shows an expanded version of the important section of the figure, involving
the first twelve frequency bands. It is seen that for the large majority of
the frequency bands, the random walk model is upheld; but the first, second
and third bands are significantly greater than the model would lead us to
expect. In fact, for all six of the major S E.C. indices (composite, manu-
facturing, transportation, utility, mining and trade, finance and services)
the second and third bands are significantly larger than would be expected
from the model and the first, fourth and fifth are consistently above the
expected spectrum. It thus seems that the model, although extremely success-
ful in explaining most of the spectral shape, does not adequately explain the
strong long run (24 months or more components of the series). Nothing definite
can be said about the business cycle component of these series as they cover too
short a time-span.

However, by using the monthly Standard and Poor common stock price
index for the period October 1875 to March 1952 and the monthly Dow-Jones stock
price index for the years 1915 to 1958, we were able to investigate the business
cycle component more closely. Figure 5 shows the estimated power spectrum of
the Standard and Poor series after an important trend in mean has been removed
by using moving averages of lengths 80 and 36. (The spectrum was estimated
at 240 frequency bands but only the first 100 are shown.) A small peak at 40
months can be seen but it is not statistically significant as a smooth curve
can easily be inserted between the 90% confidences of the estimated spectrum
in this area. Even after trend removal, this peak only accounts for slightly
less than 10% of the total remaining variance. Thus, the component corresponding
to the American business cycle of approximately 40 months, although noticeable,
is not of particular importance and is much less important than components with
periods of five years or more.
Power spectra were estimated for the Standard and Poor index over each of the periods 1875-1896, 1896-1917, 1917-1934 and 1934-1956. Although each one of these periods is too short to give definite information about the 40-month cycle, the eight-month component was invariably of some importance, this being a harmonic of the business cycle.¹)

The estimated spectra of the Dow-Jones stock price series were similar to those of the Standard and Poor index and thus provide confirmation of the accuracy of these results. The analysis of this index produces no additional information in this connection.

Certain other frequencies are also consistently outside the confidence bands but they are only those bands corresponding to a monthly variation and to the second and third harmonics of an annual (seasonal) component. In the spectrum shown, neither of the two peaks corresponding to 4 months and 1 month account for more than one-third of one percent of the total variance of the series not already explained by the random-walk hypothesis. Therefore it seems extremely unlikely that either of the "cycles" could be profitably exploited by investors. For no series so far studied has there been a significant twelve month component, although in several of them its harmonics have been just visible, yet it is commonly assumed that the New York stock market possesses a significant seasonal movement (e.g. the "summer rise"). (Further details can be found in Appendix B.)

The estimated coherence and phase diagrams between the various S.E.C. stock price indices produced few novel results. The full details are given in Appendix B but the more noteworthy results can be summarized by the following figure:

¹) It should be noted that a good clue to possibly important long-run components which, due to the shortness of the data, do not stand out in the estimated spectrum, is provided when harmonics of this component have some importance.
A strong connection (coherence generally high) is denoted by $\leftrightarrow$ and a moderately strong connection is denoted by $\longrightarrow$. The coherence diagram between the stock price indices for Manufacturing and Transportation is shown in the upper half of Fig. 7. The utilities index appears to be unrelated to the other four major subsections of the composite index shown in the table above. The indices for durables and non-durables are strongly connected, but the index for the Radio and Television section of durables have no connection with the motor manufacturers' section; neither has the rail transport index with the air transport index. In no case is there any indication of one series leading another.

Results which are more surprising arise from the coherence diagrams between certain weekly price series and the corresponding volume of sales of stocks. The lower section of Fig. 7 shows the coherence between the S.E.C. composite price index and the over-all volume of turnover of stocks of the New York Stock Exchange for the same period. The coherence is seen to be extremely low. Only at the third frequency point could one reject the hypothesis that the true coherence is zero with any worthwhile confidence. The exceptional frequency (corresponding to a period of 40 weeks) appears to correspond to no known phenomena and we suggest on the basis of our present knowledge that it is spurious. However, as spectral analysis is applied to more and more areas of the economy it is possible that other series also exhibit unexplained characteristics which, if taken together, may give rise to a new meaningful interpretation. The coherence diagrams between the
weekly closing prices and the turnover of stocks for the two companies so
considered (General Electric Corporation and Idaho Power) were also low for
all frequencies, including the third, which was large in Fig. 6.

The phase diagrams contain little worthwhile information as the
Corresponding coherences are very low, but there is a slight tendency to
oscillate about the value π. These results seem to indicate that, at least
in the short-run, and for the normal day-to-day or week-to-week workings of
the stock exchange the movements in the amount of stock sold are unconnected
with movements in price. It might be argued that the first-difference of
price should have been correlated to the volume sold; but the connection cannot
be large between these series and still give a low coherence. This statement,
of course, says nothing about exceptional price movements or of trends in
either series, it being a comment on the normal oscillations of the series.
Perhaps this result concerning the volume of sales is not surprising when
the previous result is recalled, i.e., that in the very short run prices can
be well represented by a random walk model. This follows, since in the short
run the direction of price changes cannot be predicted and so stocks cannot be
profitably sold on such predictions. We realize that professional stock ex-
change operators may from time to time have special knowledge of particular
movements and so may make a profit using this knowledge, such as when a stock-
broker influences large numbers of his customers to buy a particular stock.
But this section of the market is clearly a small one.

We finally mention some results by M. Hatanaka to be published else-
where (Granger, op. cit. Chapter 12) concerning the question of whether stock
price indices provide useful leading business cycle indicators. The coherence
diagram, between the monthly Standard and Poor Index with Industrial Production
for the period 1919-1956 and between the same index and bank clearing for 1875-1956, are both generally low. The phase-diagrams suggest that the stock price index has leads of 1-1/4 months and 2 months respectively in the relevant frequency bands.

The whole problem of studying business-cycle indicators is bedeviled with the vagueness of the concept of the business cycle movement of the economy and the huge inaccuracies known to exist in many series used to indicate the cycles (see Oskar Morgenstern: On the Accuracy of Economic Observations, Princeton University Press, 1950, a completely revised and enlarged edition being in press). Hatanaka's study did not need to use the extremely vague notion of 'turning-points' but the significance of the leads is still small, particularly as the coherences found were small, indicating that the stock price indices are poor indicators. It is as though the whole stock exchange were continually predicting the long-run movements of the economy and about 55% of these predictions were actually correct. This cannot be considered to be a very impressive record of prediction.

The results of this study show, we believe, some of the power of spectral analysis. This new statistical method is still under active development. In applications to fields other than economics it has shown that it can discover cycles of obscure phenomena hidden in a great deal of background noise. If in the case of its application to the stock market no commonly assumed strong 'cycles' have been found, this is then an indication that such cycles have no reality. 1) Once one gets used to thinking in terms of frequencies,

1) In the face of the new evidence one of us feels that he has to modify earlier statements made about the alleged existence, duration and interaction of stock market "cycles". These statements were made on the strength of the determination of such "specific cycles" by the National Bureau of Economic Research, but the existence of such cycles is not supported by the present findings. They are probably spurious. Clearly many of the consequences drawn also have to go overboard. Cf. O. Morgenstern: International Financial Transactions, op. cit.
the interpretation of the results will become rather natural for economics. The present investigation being a pilot study, and the methods used still being capable of extension, suggest that our findings may be considered to be preliminary. Yet we feel that the dominant features of stock market behavior have been illuminated.

Finally there is an interest to say a few words about the implications of our findings for investment behavior. To the extent that stock prices perform random walks the short term investor engages in a fair gamble, which is slightly better than playing roulette, since that game is biased in favor of the bank. For the long term investor, i.e., one who invests at the very minimum for a year, the problem is to identify the phases of the different long-run components of the over-all movement of the market. The evidence of "cycles" obtained in our studies is so weak that "cyclical investment" is at best only marginally worthwhile. Even this small margin will rapidly disappear as it is being made use of. The extreme weakness of the seasonal component is an example of a cycle which has been practically removed by utilizing the opportunities it seemed to offer.

1) An interesting open problem is to see whether individual stocks (rather than indices) exhibit runs.

2) That is, without considering costs of purchases and sales.
Appendix A. Some Technical Considerations

A stochastic process $X_t$ is stationary, in the wide sense, if

$$E[x_t] = m, \quad E[(X_t - m) (X_{t-\tau} - m)] = \mu_\tau \quad \text{for all } t.$$  

The power spectrum of such a series, $f(\omega)$, is defined by

$$\mu_\tau = \int_{-\pi}^{\pi} e^{i\tau\omega} f(\omega) \, d\omega.$$  

Let a set of stochastic processes $\{X_t(k)\}$ be given with a corresponding set of power spectra $\{f(\omega,k)\}$, then if a non-stationary process $Y_t$ is defined by

$$Y_t = X_t(t)$$

we may call $f(\omega,t)$ the instantaneous spectrum of $Y_t$ at time $t$. Provided that $f(\omega,t)$ changes slowly with time, it may be shown that if one estimates the spectrum of $Y_t$ as though it were a stationary series, one is actually estimating (approximately) $\frac{1}{n} \sum_{t=1}^{n} f(\omega,t)$, where $n$ is the length of data available. Thus, if a series has a slowly trending variance but no trend in mean, the effect on the estimated spectrum will merely be a scale factor.

It is for this reason that spectral methods could be used directly on the stock price series without transformations such as the logarithmic transformations. Such transformations are, of course, sensible if the utility function attached to the price of the variable being considered is proportional to the (trending) mean price. However, over the relatively short period used in the analysis, any trend in variance was not of particular importance in the method used.

Trends in mean are, of course, of greater importance, especially when the low frequencies are being particularly studied. The moving-average method
of trend removal used on the S.E.C. series was to apply two moving averages of
lengths 104 and 78 and then subtract the resulting series from the original
series. The effect is to multiply the spectrum of the series by

\[ [1 - a(\omega, 104) \cdot a(\omega, 78)]^2 \]

where

\[ a(\omega, m) = \frac{\sin \frac{m\omega}{2}}{m \sin \frac{\omega}{2}}. \]

If a series is generated by a random walk model

\[ X_t - X_{t-1} = \epsilon_t \]

where \( \epsilon_t \) is white noise, it will have power spectrum

\[ f_x(\omega) = \frac{k}{2(1 - \cos \omega)} \]

where

\[ k = \frac{\sigma^2}{2\pi} \]

We note that an alternative representation for such a series is

\[ X_t = \sum_{j=0}^{\infty} \epsilon_{t-j} \]

and thus, in theory, the variance of \( X_t \) is infinite. This also follows from
the fact that

\[ \int_{-\pi}^{\pi} f_x(\omega) d\omega = \infty. \]
However

\[ \int_{D} f_{X}(\omega) \, d\omega < \infty, \]

where \( D \) is the complete segment of the real line \((-\pi, \pi)\) excluding the narrow band \((-\delta, \delta)\) centered at zero, for any \( \delta > 0 \). Thus, if \( F[\cdot] \) is a filter which has a continuous transfer function \( a(\omega) \) such that

\[ \frac{a(\omega)}{1 - \cos \omega} \]

has a finite value at \( \omega = 0 \), then the process \( F[X_t] \) has finite variance and can be analyzed by spectral methods. The fact that we know the series \( X_t \) only over a finite length of time is essentially equivalent to such a filter.

Processes generated by the Markov model

\[ X_t - aX_{t-1} = \epsilon_t, \]

where \( \epsilon_t \) is white noise, have very similarly shaped spectra, except for very low frequencies, when the parameter \( a \) takes any value between 0.8 and 1.1. Thus, although the value \( a = 1 \), giving a random walk model, is acceptable from an economic point of view, the methods presented in the paper do not prove that this is the correct value. We thus conclude that the infinite variance aspect of the random walk model need not overly concern us.

The spectral estimating method used throughout was the Tukey-Hanning estimate as described in Granger op. cit. Chapter 4. The associated estimating lags mentioned in the following appendix are the number of frequency bands for which the spectrum was estimated. Thus, if \( m \) lags were used, the \( k^{th} \) frequency band begins at frequency \( \frac{\pi k}{m} \).
Appendix B. Description of Series Analyzed

The various stock market series analyzed for this study are listed below, together with the major results.

Power Spectra

In describing the estimated power spectra, the following abbreviations are used:

(i) **Shape of spectrum**: U = 'usual' shape, as described in section 2 above;

U* = 'usual' shape, except that some or all of the lower frequencies are less prominent than usual.

(ii) **Annual harmonics visible**: If any of the annual harmonics are visible, these are noted. \( \alpha = 12 \) months, \( \beta = 6 \) months, \( \gamma = 4 \) months, \( \delta = 3 \) months.

(iii) **Prominent frequencies**: Frequency bands which are particularly prominent, other than connected with the annual component, are listed. \( M = \) monthly component, \( E = \) eight month component.

Cross Spectra

In describing the cohenence diagrams, the following terms are used:

(i) **Very high**: Many coherence values above 0.95

(ii) **High**: Some coherence values above 0.8, most above 0.5

(iii) **Moderate**: Most coherence values between 0.6 and 0.3

(iv) **Low**: Few coherence values above 0.4

(v) **Very Low**: Majority of values below 0.25

**Phase Diagrams**: These attempt to discover whether one series is leading another. If no such lead is found the abbreviation N is used, if the diagram
contains little or no useful information (due, usually, to low coherence) the abbreviation L is used.


Trend adjustment by use of two moving averages of lengths 104, 78
\[ 1957-1959 = 100 \]
Original number of data = 1182
Number after trend adjustment = 1002
Estimating Lags = 156

<table>
<thead>
<tr>
<th>Power Spectra</th>
<th>Shape of Spectrum</th>
<th>Annual Harmonics</th>
<th>Prominent Frequencies</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Composite - 300 stocks in 32 industrial groups</td>
<td>U</td>
<td>β</td>
<td>M,E</td>
</tr>
<tr>
<td>2. Manufacturing</td>
<td>U</td>
<td>β, γ</td>
<td>M,E</td>
</tr>
<tr>
<td>3. Durable Goods Manufacturing</td>
<td>U</td>
<td>γ</td>
<td>M,E</td>
</tr>
<tr>
<td>4. Radio, Television and Communications Equipment</td>
<td>U</td>
<td>-</td>
<td>M,E</td>
</tr>
<tr>
<td>5. Motor Vehicle Manufacturing</td>
<td>U</td>
<td>γ,δ</td>
<td>M</td>
</tr>
<tr>
<td>7. Transportation</td>
<td>U</td>
<td>-</td>
<td>M,E</td>
</tr>
<tr>
<td>8. Railroad</td>
<td>U</td>
<td>-</td>
<td>M,E</td>
</tr>
<tr>
<td>9. Air Transportation</td>
<td>U</td>
<td>-</td>
<td>E</td>
</tr>
<tr>
<td>10. Utilities</td>
<td>U</td>
<td>δ</td>
<td>-</td>
</tr>
<tr>
<td>11. Trade, Finance and Service</td>
<td>U</td>
<td>-</td>
<td>M</td>
</tr>
<tr>
<td>12. Mining</td>
<td>U</td>
<td>-</td>
<td>M,E</td>
</tr>
</tbody>
</table>
Cross Spectra

<table>
<thead>
<tr>
<th>Composite Index with</th>
<th>Manufacturing</th>
<th>Transportation</th>
<th>Utilities</th>
<th>Trade</th>
<th>Mining</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Manufacturing with</td>
<td>Durable</td>
<td>Non-Durable</td>
<td>Transportation</td>
<td>Utilities</td>
<td>Trade, Finance and Service</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Mining</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Durable with</td>
<td>Non-Durable</td>
<td>Radio, Television and Communication</td>
<td>Equipment</td>
<td>Motor Vehicle</td>
<td></td>
</tr>
<tr>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Radio, Television and Communication Equipment with</td>
<td>Motor Vehicle</td>
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</tr>
<tr>
<td>Transportation with</td>
<td>Railroad</td>
<td>Air Transportation</td>
<td>Mining</td>
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<tr>
<td>Railroad with</td>
<td>Air Transportation</td>
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<td>Utilities with</td>
<td>Mining</td>
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<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Trade, Finance and Service with</td>
<td>Mining</td>
<td></td>
<td></td>
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</table>

Coherence    Phase

<table>
<thead>
<tr>
<th></th>
<th>Very High</th>
<th>High</th>
<th>Moderate</th>
<th>Low</th>
<th>Moderate to High</th>
<th>Low</th>
<th>Moderate</th>
<th>N</th>
<th>Moderate</th>
<th>N</th>
<th>High</th>
<th>N</th>
</tr>
</thead>
</table>

II. Stock Prices of Six Large U.S. Companies, Monthly: January 1946 - December 1960

Average of low and high prices for each month
Original number of data = 180
Estimating lags = 60

Power Spectra

<table>
<thead>
<tr>
<th></th>
<th>Annual Harmonics</th>
<th>Prominent Frequencies</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shape</td>
<td>Visible</td>
<td>Frequencies</td>
</tr>
<tr>
<td>-------</td>
<td>---------</td>
<td>-------------</td>
</tr>
<tr>
<td>1. American Tobacco</td>
<td>U</td>
<td>-</td>
</tr>
<tr>
<td>2. General Foods</td>
<td>U</td>
<td>8</td>
</tr>
<tr>
<td>3. American Can</td>
<td>U*</td>
<td>8</td>
</tr>
<tr>
<td>4. Woolworth</td>
<td>U</td>
<td>8</td>
</tr>
<tr>
<td>5. Chrysler</td>
<td>U</td>
<td>-</td>
</tr>
<tr>
<td>6. U. S. Steel</td>
<td>U*</td>
<td>-</td>
</tr>
</tbody>
</table>
Cross Spectra

<table>
<thead>
<tr>
<th></th>
<th>Coherence</th>
<th>Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>American Tobacco with</td>
<td>High</td>
<td>N</td>
</tr>
<tr>
<td>General Foods</td>
<td>Mixed</td>
<td>U</td>
</tr>
<tr>
<td>American Can</td>
<td>High</td>
<td>N</td>
</tr>
<tr>
<td>Woolworth</td>
<td>High</td>
<td>N</td>
</tr>
<tr>
<td>Chrysler</td>
<td>Moderate</td>
<td>N</td>
</tr>
<tr>
<td>U. S. Steel</td>
<td>High</td>
<td>N</td>
</tr>
<tr>
<td>General Foods with</td>
<td>Mixed</td>
<td>L</td>
</tr>
<tr>
<td>American Can</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Woolworth</td>
<td>High</td>
<td>N</td>
</tr>
<tr>
<td>Chrysler</td>
<td>High</td>
<td>N</td>
</tr>
<tr>
<td>U. S. Steel</td>
<td>High</td>
<td>N</td>
</tr>
<tr>
<td>American Can with</td>
<td>Mixed</td>
<td>L</td>
</tr>
<tr>
<td>Woolworth</td>
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<td></td>
</tr>
<tr>
<td>Chrysler</td>
<td>Mixed</td>
<td>L</td>
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<tr>
<td>U. S. Steel</td>
<td>Mixed</td>
<td>L</td>
</tr>
<tr>
<td>Woolworth with</td>
<td>Mixed</td>
<td>L</td>
</tr>
<tr>
<td>Chrysler</td>
<td></td>
<td></td>
</tr>
<tr>
<td>U. S. Steel</td>
<td>Mixed</td>
<td>N</td>
</tr>
<tr>
<td>Chrysler with</td>
<td>Mixed</td>
<td>N</td>
</tr>
<tr>
<td>U. S. Steel</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

III. Standard and Poor Common Stock Price Index, Industrials, Rails and Utilities, Monthly, 1875-1956

Industrial production index, monthly, 1919-1956
Trend adjusted by moving averages of lengths 80, 36
Estimating lags = 60, except where specified

Power Spectra

<table>
<thead>
<tr>
<th></th>
<th>Shape</th>
<th>Annual Harmonics</th>
<th>Prominent Frequencies</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. October 1875 - March 1952</td>
<td>U</td>
<td>β</td>
<td>Slight 40 Month, E</td>
</tr>
<tr>
<td>(estimating lags: 240)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. October 1875 - October 1896</td>
<td>U</td>
<td>β,γ</td>
<td>Slight 10 month</td>
</tr>
<tr>
<td>3. October 1875 - October 1917</td>
<td>U</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>4. October 1896 - October 1917</td>
<td>U</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>5. October 1917 - March 1952</td>
<td>U</td>
<td>β</td>
<td>E</td>
</tr>
<tr>
<td>6. October 1917 - March 1934</td>
<td>U*</td>
<td>β</td>
<td>E</td>
</tr>
<tr>
<td>(estimating lags: 40)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7. March 1934 - March 1952</td>
<td>U</td>
<td>-</td>
<td>E</td>
</tr>
<tr>
<td>(estimating lags: 40)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8. Industrial Production Index</td>
<td>U</td>
<td>α,β,γ,σ</td>
<td>-</td>
</tr>
<tr>
<td>(estimating lags:120)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Cross Spectra

Estimating lags = 120

<table>
<thead>
<tr>
<th></th>
<th>Coherence</th>
<th>Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Standard and Poor Index with</td>
<td>Low with</td>
<td>Some evidence of a</td>
</tr>
<tr>
<td>Industrial Production Index</td>
<td>occasional</td>
<td>1-1/4 month lead in low frequencies.</td>
</tr>
<tr>
<td>1919-1956</td>
<td>moderates.</td>
<td></td>
</tr>
<tr>
<td>2. Standard and Poor Index with</td>
<td>As above</td>
<td>As above, but lead</td>
</tr>
<tr>
<td>Bank Clearings, 1875 - 1956</td>
<td></td>
<td>of 2 months.</td>
</tr>
<tr>
<td>(Stock prices lead in both cases.)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
IV. Dow-Jones Industrial Stock Price Index, Monthly, January 1915 - June 1961
(20 stocks through September 1928, 30 thereafter)

Power Spectra

1. 1915-1958 (estimating lags: 120) Shape Annual Prominent
   Harmonics Frequencies
   U - E
2. 1915-1934 (estimating lags: 60)
   U β -
3. 1935-1957 (estimating lags: 60)
   U - E

V. Price and Volume Data

Weekly data, January 1957 to December 1961
S.E.C. composite price index, over-all volume of sales during week:
Closing price and volume of sales of stocks of two firms: General Electric
Corporation, Idaho Power.

Power Spectra

1. Composite price index Shape Annual Prominent
   U Harmonics Frequencies
   U* β M
2. Over-all volume series
   U - -
3. G.E.C. closing prices
   U* β M
4. Volume of G.E.C. stocks sold
   U - -
5. Idaho Power closing prices
   Special shape requiring further investigation
6. Volume of Idaho Power stocks sold

Cross Spectra

1. Composite price index with over-all volume series Coherence Phase
3. Idaho Power closing prices with Idaho Power Very Low L