THE ESTIMATION OF COBB-DOUGLAS TYPE FUNCTIONS WITH MULTIPLICATIVE AND ADDITIVE ERRORS: A FURTHER ANALYSIS

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1. Introduction

In a recent article in this journal Goldfeld and Quandt (4) suggested a maximum likelihood approach to the estimation of a Cobb-Douglas type model when the model includes both multiplicative and additive disturbance terms. As expected, an analytical expression for the solution to the maximization problem did not exist. Indeed, because of the complexity of the likelihood function, their maximization algorithm had to be used in conjunction with a numerical integration technique.

The purpose of this paper is to generalize and simplify the work by Goldfeld and Quandt. Specifically, an estimation technique is suggested which does not require the specification of the disturbance terms beyond their means and variances, which does not require the compounding of a maximization algorithm with a numerical integral technique, but yet leads to asymptotically efficient estimates of the parameters of the regression function. In addition, the procedure readily lends itself to interpretation. For instance, it will become evident that if the distribution of the multiplicative disturbance term is not known, the scale parameter of the model (unlike the other parameters) will not be identified.

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2. Model Specification and Estimation

Consider the model suggested by Goldfeld and Quandt,

\[(1) \quad y_t = \alpha_0 x_{1t} \ldots x_{kt} e_t + v_t, \quad t = 1, \ldots, n,\]

where \(x_{1t} \ldots x_{kt}\) are the \(t^{th}\) observations on the \(k\) independent variables, \(e_t\) and \(v_t\) are the \(t^{th}\) values of the disturbance terms. We assume that \(u_t\) and \(v_t\) are independent of each other, are independent of \(x_{1t} \ldots x_{kt}\), and \(E u_t = E v_t = 0\), and \(E u_t^2 = \sigma_u^2\), and \(E v_t^2 = \sigma_v^2\). We also assume that \(u_t\) and \(v_t\) are independent of \(u_s\) and \(v_s\) for all \(s \neq t\). We do not, however, assume normality on any other particular distribution.

Let \(A\) be the mean of \(e_t\). Then \(e_t\) may be expressed as

\[(2) \quad e_t = A + \phi_t,\]

where \(E \phi_t = 0\). Substituting (2) into (1) we obtain

\[(3) \quad y_t = B x_{1t} \ldots x_{kt} + \omega_t,\]

where \(B = \alpha_0 A\), and \(\omega_t = \alpha_0 x_{1t} \ldots x_{kt} \phi_t + v_t\). Since \(\phi_t\) and \(v_t\) are independent of \(x_t = (x_{1t} \ldots x_{kt})\), it follows that \(E[\omega_t | x_t] = 0\). Therefore if we ignore the heteroscedasticity of \(\omega_t\) and apply nonlinear least squares to (3), the resulting estimates of \(B\) and \(\alpha_1, \ldots, \alpha_k\), although not efficient, will be consistent.\(^2\) The remainder of the procedure may now be evident. These estimates will be used to obtain a consistent estimate of the variance of \(\omega_t\), then (3) will be transformed to rid \(\omega_t\) of its heteroscedasticity, and finally, nonlinear least

\(^1\) Goldfeld and Quandt assume that the independent variables are nonstochastic. This assumption, however, is unnecessary for the results of this paper.

\(^2\) The proof of this would be almost identical to the one described in Aigner and Goldberger (1, p. 715). Actually, in a different context, such a proof is outlined below.
squares will be applied to the transformed model.

The conditional variance of \( w_t \) is easily shown to be

\[
\begin{align*}
E[w_t^2 | x_t] &= (\alpha_0^2 \phi^2) x_{1t} \ldots x_{kt} + \sigma_v^2 = cz_t + \sigma_v^2
\end{align*}
\]

where \( \sigma_\phi^2 \) is the variance of \( \phi_t \), \( c = \alpha_0^2 \phi^2 \), and \( z_t = x_{1t} \ldots x_{kt} \).

Now, the consistent estimates of \( B \) and \( a_1, \ldots, a_k \) described above enable us to obtain a consistent estimate of \( w_t \) from (3), namely

\[
\begin{align*}
\hat{w}_t &= y_t - \hat{\beta} x_{1t} \ldots x_{kt} \\
\hat{w}_t^2 &= \hat{w}_t^2 + \Delta_t
\end{align*}
\]

where the probability limit of \( \Delta_t \) with respect to \( n \) is zero. Given (6) and (4) it is clear that a consistent estimate of the conditional variance of \( w_t \) can be easily obtained by ordinary least squares via the linear regression:

\[
\begin{align*}
\hat{w}_t^2 &= \sigma_v^2 + cz_t + \varepsilon_t
\end{align*}
\]

where \( z_t = x_{1t} \ldots x_{kt} \), and \( \varepsilon_t \) is a disturbance term. Specifically, if the estimates of \( \sigma_v^2 \) and \( c \) are \( \hat{\sigma}_v^2 \) and \( \hat{c} \), then our consistent estimate of the conditional variance of \( w_t \) is

\[
\hat{\sigma}_w^2 = \hat{\sigma}_v^2 + \hat{c} z_t^2
\]

Finally, nonlinear least squares can now be applied to our basic model (3) after it has been transformed by dividing it though by \( \hat{w}_t \).

The resulting estimates of \( B \) and \( a_1, \ldots, a_k \) are obviously consistent, and, since \( \hat{\sigma}_w t \) is a consistent estimate of \( \sigma_w^2 \), these estimates should

\[\text{To see this note that (4) implies that } w_t^2 = cz_t + \sigma_v^2 + r_t, \text{ where } E[r_t^2 | x_t] = 0. \text{ Thus } \hat{w}_t^2 = cz_t + \sigma_v^2 + (r_t + \Delta_t). \text{ Since } z_t \text{ converges to } z_t \text{ the results from (7) follow. The reader should note that limits of expectations are not being taken. That convergence in probability need not imply convergence in moments is nicely demonstrated in Dhrymes (2, pp. 88-89).}\]
also be asymptotically efficient.\(^4\)

To see this let \( y, f, \) and \( w \) be the \( n \times 1 \) vectors whose \( t^{th} \)
elements are \( y_t, B x_{1t}, \ldots, x_{kt}, \) and \( v_t \). Then the proposed estimates
can be obtained by minimizing \( S \) where

\[
(9) \quad S = (y - f)' \hat{\Sigma}^{-1}(y - f)
\]

where \( \hat{\Sigma}^{-1} \) is the \( n \times n \) diagonal matrix whose \( t^{th} \) diagonal element is
\( \hat{\sigma}_{wt}^2 \). Let \( P \) denote the \((k+1) \times 1\) vector of parameters: \( P' = (B \ a_{1} \ldots a_{k}) \).

Then minimizing \( S \) with respect to \( P \) yields

\[
(10) \quad f' \hat{\Sigma}^{-1} \frac{\partial f}{\partial P} = 0
\]

where \( f' \) is the \( n \times k + 1 \) matrix \( \frac{\partial f}{\partial P} \) evaluated at our estimate \( \hat{P} \),
and \( f \) is the \( n \times 1 \) vector, \( f \), evaluated at \( \hat{P} \). Linearizing (10) about the
true parameters we have

\[
(11) \quad f' \hat{\Sigma}^{-1} (f + f' \frac{\partial f}{\partial P} (\hat{P} - P) - y) = 0
\]

or

\[
(12) \quad \hat{P} - P = (f' \hat{\Sigma}^{-1} f')^{-1} f' \hat{\Sigma}^{-1} w,
\]

where \( \Sigma \) is the \( n \times n \) diagonal matrix whose \( t^{th} \) diagonal element
is \( \sigma_{wt}^2 \). It follows from (12) that \( \hat{P} \) is consistent. Further, under
general conditions, it can be shown\(^5\) that \( \sqrt{T} (\hat{P} - P) \) convergences
in distribution to a multivariate normal with mean vector zero and

\[^4\] The estimate of the scale parameter \( \sigma_0 \) can be obtained from the
final estimate of \( B = \sigma_0 A \), and our estimate of \( c = \alpha_0^2 \sigma_0^2 \). It is
interesting to note that to do this (solve for \( \sigma_0 \)) the density of \( u_t \)
must be such that \( \sigma_0^2 \) and \( A \) depend upon a single common parameter.
Furthermore, to know how \( \sigma_0^2 \) and \( A \) relate to each other in terms of that
parameter the density of \( u_t \) must be specified.

\[^5\] See Dhrymes (2, p. 108). For a concise discussion of some of
the problems involved in discussing sequences of random variables see
Chapter 3 of Dhrymes (2).
conditional covariance matrix

\[(13) \quad V_{\hat{P}} = T(\hat{r}_p \Sigma^{-1} \hat{r}_p)^{-1}.\]

A consistent estimate of \( V_{\hat{P}} \) is obviously

\[(14) \quad \hat{V}_{\hat{P}} = T(\hat{r}_p \hat{\Sigma}^{-1} \hat{r}_p)^{-1}.\]

Since \( V_{\hat{P}} \) is the asymptotic covariance matrix of the Aitken estimator, it follows that \( \hat{P} \) is asymptotically efficient.\(^6\)

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\(^6\)One could, of course, iterate on this technique. That is, \( \hat{P} \) could be used to obtain another estimate of \( \Sigma \) which would lead to another estimate of \( P \), etc. The asymptotic properties, however, of the resulting estimate of \( P \) would not be different from those of \( \hat{P} \) described above.


